SOME EFFECTS RELATED TO THE FINITE VELOCITY OF PROPAGATION OF HEAT

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The hyperbolic heat-conduction equation is solved for periodic boundary conditions of the second kind. The amplitude, phase, and damping factor of the temperature oscillations turn out to depend on the relaxation time. The possibility of determining the velocity of propagation of heat experimentally is analyzed.

The finite velocity of propagation of heat is taken into account in the hyperbolic heat-conduction equation [1-3] and is given by

$$v = \sqrt{a/\tau_r} \tag{1}$$

This formula involves the thermal diffusivity a, which is a typical macroscopic characteristic of a material, and the relaxation time τ_r, which is a microscopic transport equation, and it is far from clear a priori whether the quantity τ_r introduced into it is to be identified with the microscopic relaxation time (let us say, the time to establish local thermodynamic equilibrium), although these quantities are certainly related.

Therefore it is clear that the experimental investigation of effects related to the finite velocity of propagation of heat is of fundamental importance. It would enable us to obtain very valuable information on thermal relaxation and on the relation of the microscopic and macroscopic characteristics of a material, and would reveal the deeper nature of thermal phenomena.

The conditions under which one should expect a manifestation of these effects can be determined from the modified heat-conduction equation [1]

$$-\lambda \frac{\partial t}{\partial x} = q + \tau_r \frac{\partial q}{\partial \tau}. \tag{2}$$

These effects will be important (in comparison with the ordinary Fourier law) and observable when the second term on the right-hand side of Eq. (2) is comparable with or larger than the first, i.e., for media with long relaxation times, e.g. rarefied gases, or for large heating or cooling rates. If the medium under investigation is a solid it is meaningful to speak of the second case. Therefore it is of interest to examine the solution of the hyperbolic heat-conduction equation for large values of $\partial q/\partial \tau$. This can be realized in practice by employing an intense laser beam whose intensity is modulated with a rather high frequency. The periodic component of the radiant flux ensures a large value of $\partial q/\partial \tau$.

Let us consider the idealized case of a semibounded space and a pure harmonic time dependence of the heat flux

$$q = q_0 \cos(\omega\tau - \varphi). \tag{3}$$

The mathematical formulation of the problem is the following:

$$\frac{\partial t}{\partial \tau} + \tau_r \frac{\partial^2 t}{\partial \tau^2} = a \frac{\partial^2 t}{\partial x^2}, \quad x \geqslant 0; \quad \tau > 0; \tag{4}$$

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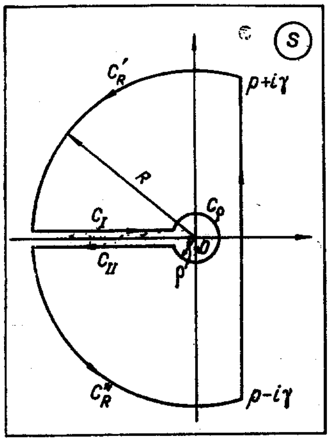

Fig. 1. Contour for evaluating the inverse Laplace transform of a function having branch points at s = 0 and s = ∞.

$$\frac{\partial t(0,\ \tau)}{\partial x} = -\frac{q_0}{\lambda}\sqrt{1+\omega^2\tau_r^2}\,[\cos\omega\tau\cos(\varphi-\psi)+\sin\omega\tau\sin(\varphi-\psi)]; \tag{5}$$

$$\frac{\partial t(\infty,\ \tau)}{\partial x} = 0; \tag{6}$$

$$\frac{\partial t(x,\ 0)}{\partial \tau} = 0; \tag{7}$$

$$t(x,\ 0) = 0; \tag{8}$$

where

$$\psi = \operatorname{arctg}\omega\tau_r \tag{9}$$

Boundary condition (5) is taken in this form on the basis of Eq. (2); as $\tau_r \to 0$ this condition goes over into the usual expression

$$\frac{\partial t(0,\ \tau)}{\partial x} = -\frac{q}{\lambda}. \tag{10}$$

Taking the Laplace transform with respect to the time and omitting the intermediate calculations we obtain for the transform

$$T(x,\ s) = \frac{q_0\sqrt{a}}{\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}\,\frac{s\cos(\varphi-\psi)+\omega\sin(\varphi-\psi)}{s^2+\omega^2}$$

$$\times\frac{1}{\sqrt{s^2+\frac{s}{\tau_r}}}\exp\left[-\sqrt{\frac{\tau_r}{a}\left(s^2+\frac{s}{\tau_r}\right)}\,x\right]. \tag{11}$$

Using a table of transforms [4] and the Borel formula (the convolution theorem) we obtain the solution of the original problem (4)-(8) in the form

$$t(x,\ \tau) = 0 \quad \text{for} \quad 0<\tau\leqslant\tau_r = x\sqrt{\frac{\tau_r}{a}};$$

$$t(x,\ \tau) = \frac{2q_0\sqrt{a}}{\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}\cos\left(\varphi-\psi-\frac{\pi}{4}\right)$$

$$\times\int\limits_{\tau_d}^{\tau}\cos\left(\omega\eta-\frac{\pi}{4}\right)\exp\left(-\frac{\tau-\eta}{2\tau_r}\right)I_0\left(\frac{\sqrt{(\tau-\eta)^2-\tau_d^2}}{2\tau_r}\right)d\eta \tag{12}$$

$$\text{for} \quad \tau>\tau_d.$$

It should be noted, however, that the solution in the form (12) is inconvenient for analysis. It includes both the transient and the steady-state parts of the process, and they cannot be separated. Therefore we try to find a solution in a second form by starting directly from the inversion theorem and using the theory of residues. The transform (11) satisfies all the conditions of the inversion theorem and permits the selection of a single-valued branch. It has two simple poles at $s = \pm i\omega$, and branch points at $s = 0$ and $s = \infty$. The contour of integration is shown in Fig. 1.

We have

$$2\pi i\sum_{k=1}^{2}\operatorname{Res}[T(x,\ s)\exp(s\tau);\ \pm i\omega] = \oint\limits_{C} T(x,\ s)\exp(s\tau)\,ds, \tag{13}$$

where $C = C_\rho + C'_R + C_I + C_\rho + C_{II} + C''_R$ (cf. Fig. 1).

After finding the residues the left-hand side of Eq. (13) becomes

$$\sum_{k=1}^{2}\operatorname{Res}[T(x,\ s)\exp(s\tau);\ \pm i\omega] = \frac{q_0\sqrt{a}}{2\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}$$

$$\times\left\{\frac{\exp\left[-i(\omega\tau-\varphi+\psi)-\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2-i\frac{\omega}{\tau_r}}\,x\right]}{\sqrt{-\omega^2-i\frac{\omega}{\tau_r}}}+\frac{\exp\left[i(\omega\tau-\varphi+\psi)-\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}\,x\right]}{\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}}\right\}. \quad (14)$$

Let us consider the right-hand side of (13). If the radius of the larger circle tends to infinity $R \to \infty$, by Jordan's lemma the integrals $\int_{C_R'} T(x, s)\exp(s\tau)ds$ and $\int_{C_R''} T(x, s)\exp(s\tau)ds$ vanish as $\rho \to 0$; the integral $\int_{C_\rho} T(x, s)\exp(s\tau)ds$ also vanishes. On the contour C_I we take $s = r\exp(i\pi)$, and on $C_{II} = s = r\exp(-i\pi)$. In the limit as $R \to \infty$ and $\rho \to 0$, taking account of the value of the square root $\sqrt{s^2 + s/\tau_r}$, the integrals along C_I and C_{II} can be written in the form

$$\int_{C_I} T(x, s)\exp(s\tau)\,ds + \int_{C_{II}} T(x, s)\exp(s\tau)\,ds = -\frac{2q_0\sqrt{a}}{i\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}$$

$$\times\int_0^\infty \frac{-r\cos(\varphi-\psi)+\omega\sin(\varphi-\psi)}{r^2+\omega^2}\,\frac{\cos\left(x\sqrt{\frac{\tau_r}{a}\left(r^2-\frac{r}{\tau_r}\right)}\right)}{\sqrt{r^2-\frac{r}{\tau_r}}}\exp(-r\tau)\,dr. \quad (15)$$

Equation (14) gives the steady-state solution of the problem, and (15) the transient behavior. It is clear directly that (15) vanishes as $\tau \to \infty$.

Thus substituting (14) and (15) into (13) and letting $R \to \infty$ and $\rho \to 0$ we obtain the solution of the problem (4)-(8):

$$t(x, \tau) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} T(x, s)\exp(s\tau)\,ds$$

$$= \frac{q_0\sqrt{a}}{2\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}\left\{\frac{\exp\left[-i(\omega\tau-\varphi+\psi)-x\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2-i\frac{\omega}{\tau_r}}\right]}{\sqrt{-\omega^2-i\frac{\omega}{\tau_p}}}\right.$$

$$+\frac{\exp\left[i(\omega\tau-\varphi+\psi)-x\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}\right]}{\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}}$$

$$\left.-\frac{2}{\pi}\int_0^\infty \frac{-r\cos(\varphi-\psi)+\omega\sin(\varphi-\psi)}{r^2+\omega^2}\,\frac{\cos\left(x\sqrt{\frac{\tau_r}{a}\left(r^2-\frac{r}{\tau_r}\right)}\right)}{\sqrt{r^2-\frac{r}{\tau_r}}}\exp(-r\tau)\,dr\right\}. \quad (16)$$

Let us consider the steady-state part of the solution

$$t_1(x, \tau) = \frac{q_0\sqrt{a}}{2\lambda\sqrt{\tau_r}}\sqrt{1+\omega^2\tau_r^2}\left\{\frac{\exp\left[i(\omega\tau-\varphi+\psi)-x\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2-i\frac{\omega}{\tau_r}}\right]}{\sqrt{-\omega^2-i\frac{\omega}{\tau_r}}}\right.$$

$$\left.+\frac{\exp\left[i(\omega\tau-\varphi+\psi)-x\sqrt{\frac{\tau_r}{a}}\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}\right]}{\sqrt{-\omega^2+i\frac{\omega}{\tau_r}}}\right\}. \quad (17)$$

It is clear that the second term in (17) is the complex conjugate of the first. After separating the real and imaginary parts in both terms we obtain the final expression for the steady-state process

$$t_1(x,\ \tau)=\frac{q_0\sqrt{a}}{\lambda\sqrt{\omega}}\sqrt[4]{1+\omega^2\tau_r^2}\exp\left[-\sqrt{\frac{\omega}{2a}\left(-\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)x}\right]$$

$$\times\cos\left[\omega\tau-\varphi-\operatorname{arctg}\left(-\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)-\sqrt{\frac{\omega}{2a}\left(\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)x}\right]. \tag{18}$$

As $\tau_r \to 0$

$$t(x,\ \tau)=\frac{q_0\sqrt{a}}{\lambda\sqrt{\omega}}\exp\left(-x\sqrt{\frac{\omega}{2a}}\right)\cos\left(\omega\tau-\varphi-\frac{\pi}{4}-x\sqrt{\frac{\omega}{2a}}\right), \tag{19}$$

which agrees with the solution of the parabolic heat-conduction equation with a similar boundary condition [5].

If we set $\psi = 0$ in (18) and divide it by $\sqrt{1 + \omega\tau_r^2}$ it is seen from (5) that we obtain the solution of the hyperbolic heat-conduction equation for the usual boundary conditions (10):

$$t(x,\ \tau)=\frac{q_0\sqrt{a}}{\lambda\sqrt{\omega}\sqrt[4]{1+\omega^2\tau_r^2}}\exp\left[-\sqrt{\frac{\omega}{2a}\left(-\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)x}\right]$$

$$\times\cos\left[\omega\tau-\varphi-\operatorname{arctg}\left(\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)-\sqrt{\frac{\omega}{2a}\left(\omega\tau_r+\sqrt{1+\omega^2\tau_r^2}\right)x}\right]. \tag{20}$$

A comparison of Eqs. (18) or (20) with (19) shows that taking account of the finite velocity of propagation of heat leads to changes in the amplitude and phase of the temperature oscillations in a medium and on its surface. The damping factor of the temperature oscillations is changed also.

As the frequency increases the damping increases more slowly than in the solution of the parabolic equation. It is particularly interesting that as $\omega \to \infty$ the damping factor does not tend to ∞ but to the finite value

$$k=\frac{1}{2\sqrt{a\tau_r}}, \tag{21}$$

as is easy to show by calculating the limit of the exponential in (18).

It is interesting also that as the frequency increases, the amplitude of the temperature oscillations does not approach zero as in (19). For example, as $\omega \to \infty$ the amplitude of the temperature oscillations of the surface ($x = 0$) is seen from (18) to be

$$A=\frac{q_0\sqrt{a}}{\lambda}\sqrt{\tau_r} \tag{22}$$

This last result is a consequence of the boundary condition (5); in (20) the amplitude tends to zero as $\omega \to \infty$.

The effects obtained are not unexpected from the physical point of view. Thus it is clear physically that for large values of the relaxation time the transport of thermal excitation inside the body will be slowed down; this leads to an increase in the amplitude of the temperature oscillations on the surface for periodic heating. As $\tau_r \to \infty$ the velocity of propagation of heat tends to zero and the incoming energy piles up at the surface of the body and the amplitude of the temperature oscillations must consequently increase without bound. This result, incidently, shows that in solving the hyperbolic heat-conduction equation the boundary condition must be taken in the form (5).

The effect of the finite velocity of propagation of heat on the amplitude, damping factor, and phase of the temperature oscillations can be used to determine the velocity or the relaxation time experimentally. One possible method consists in investigating and recording the thermoacoustic effect. It is known [6] that a periodic temperature distribution will produce thermal stresses which in turn generate elastic vibrations in the body under study. The amplitude and phase of these vibrations will obviously be related to the amplitude, phase, and damping factor of the temperature oscillations. The elastic vibrations will propagate in the body and can be received and recorded.

If a periodic thermal flux is supplied locally, for example, in the form of a circular spot from a laser beam, then in addition to volume oscillations Rayleigh surface waves [7] are produced and propagate from the spot. These waves also can be received and recorded. The parameters of the thermoelastic oscillations can be calculated rigorously from the solution of the thermoacoustic equations; this is an independent problem.

NOTATION

V	is the velocity of propagation of heat;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
τ_r	is the relaxation time;
τ_d	is the delay time;
q	is the specific heat flux incident on the surface of the body;
ω	is the angular frequency of oscillations;
φ	is the initial phase of the oscillations;
I_0	is the zero order Bessel function of imaginary argument.

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